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ON EUCLID'S FIRST THREE POSTULATES

John A. Fossa
Universidade Federal do Rio Grande no Norte – UFRN – Brasil

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Abstract

An analysis of the first three postulates of Euclid's *Elements* indicates that the *Elements* may be a theorizing of surveying, an activity understood as being involved in the creation of the cosmos, and, thus, mathematics may be understood as mimesis of the divine.

Keywords: Mathematics and Philosophy, History of Geometry, Euclid, Postulates.

[SOBRE OS TRÊS PRIMEIROS POSTULADOS DE EUCLIDES]

Resumo

Uma análise dos primeiros três postulados dos *Elementos* de Euclides indica que os próprios *Elementos* poderão ser uma teorização da atividade de agrimensura, sendo isto entendido como envolvido com a criação do cosmos, e, assim, a matemática pode ser entendida como uma mimese do divino.

Palavras-chave: Matemática e Filosofia, História da Geometria, Euclides. Postulados.

Introduction

It is well known that the first three of Euclid's postulates are supposed to stipulate that geometric constructions using the ruler (that is, an unmarked straightedge) and the (collapsible) compass are legitimate activities in Euclidean Geometry. These three postulates are given here in Heath's translation[1] (EUCLID, 1956, v. I, p. 154):

> Let the following be postulated:
> 1. To draw a straight line from any point to any point.
> 2. To produce a finite straight line continuously in a straight line.
> 3. To describe a circle with any center and distance.

It may be recalled, however, that the ancient Greeks also used other methods in specifying geometrical objects. Thus, alongside of the "point construction" of a segment (equilateral triangle, tetrahedron) by the specification of endpoints (vertices), there also existed a "fluxion construction" of a segment (square, cube) by the motion of a point (segment, square)[2]. Further, a few curves, such as the quadratrix of Hippias and the spiral of Archimedes, were specified by composite motions. Hence, the question that immediately comes to mind is this: Wherefore ruler and compass?

One answer to this question is that it was a restriction imposed by Plato on geometers in order to guarantee that their arguments would not be tainted by empirical considerations and, thus, lose their mathematical rigor. This was the answer proposed by Hermann Hankel (1874), based on some comments by Pappus. Another, proposed by Heath (1981) and subscribed to by Walter Burkert (1972) makes the 4th century geometer Oenopides of Chios the originator of the stipulation.

Nonetheless, whatever the origin of the ruler and compass restriction may be, it is not explicitly given in Euclid's postulates. These, in fact, only require that geometric constructions be built up from line segments and circular arcs. Hence, the use of ruler and compass is best seen as one way of implementing the restriction to line segments and circles. Another way of doing the same thing is by using pegs and cords, since a cord stretched between two pegs delineates the line segment between the pegs and swinging one of the pegs about the other, while maintaining the cord taut[3], describes the circle whose center is at the fixed peg and whose radius is the length of the cord. This possibility was noticed by A. Seidenberg (1959)[4].

Thesis

Considering, therefore, that a very common[5] way in which surveying was carried out in ancient cultures was by the use of stretched ropes, it would seem that Euclid's *Elements* may be considered as a theorizing of the ancient activity of surveying. Naturally, the activity of surveying, itself, was an approximate science based on rough and ready

[1] Heath's translation contains, in his notes on the text, the Greek version of the postulates. For a newer translation, see Euclides (2009).
[2] See Erickson and Fossa (1996).
[3] It is also necessary to swing the cord in such a way that is does not wrap itself about the fixed peg.
[4] See also, Seidenberg (1961).
[5] See, for example, Joseph W. Dauben (1992).

methods, but the mathematical model of this activity was seen as an exact science which furnishes absolute truth. Surveying would merit this theoretical treatment because, as the determination of limits, it would be part of the divine creation of the *kósmos* (beautiful construction) out of the primeval chaos, as discussed, for example, by Plato in the *Timaeus*. This knowledge is accessible to man because the divine demiurge is a rational being who acts in a rational manner. In fact, by doing the theoretical activity of geometry, man is engaging in a mimesis of the divine.

Considerations

The thesis set forth in the previous paragraph links mathematics, surveying and theological cosmogony in a way that, although typical of ancient thought, is not entirely congenial to the modern mind. I will, therefore, in what follows, make a series of numbered remarks that will hopefully clarify the proposed thesis.

§1. Why "surveying"?

As is well known, "geometry", etymologically speaking, signifies "measuring the Earth". "Surveying" is perhaps the closest modern equivalent. We should also take into consideration the names of the other mathematical sciences in antiquity. Thus, "astronomy" is the "measurement of the stars", by which is meant, primordially, the determination of the positions and orbits of the heavenly bodies. "Arithmetic" is the "art of counting" and the study of ratio and proportion is called "music" or "harmony". Consequently, the mathematical sciences – the very height of abstraction and certain knowledge (in the Greek view) – were all named with regard to their most conspicuous applications. Obviously, however, the point of view of the mathematicians engaged in these studies transcends that of their practical applications.[6]

§2. Is surveying co-extensive with Egyptian "cord stretching"?

The ancient Egyptians used cords to reform the property lines that were obliterated by the annual floodwaters of the Nile. The correct determination of these properties was of immense importance to the State, since they were involved in their taxation schemes and, consequently, they were invested with divine authority. But measurements using stretched cords (or, alternatively, measuring rods) were used in virtually all ancient cultures, not only to determine property rights, but also in town planning, the design of public spaces and the construction of both public and private buildings (architecture). Many of these activities were connected with sacred mathematics in a very intimate way. The classical architect Vitruvius (fl. c. 14 B.C.), for example, was interested in constructions having proportions

[6] In light of the thesis of the present note, the immediate suggestion would be that each of the mathematical sciences is a theorizing of its respective application. I believe that this is entirely correct, but, herein, will limit myself to the case of geometry, though I will return to this in the conclusion.

that would embody the analogy between the macrocosm and the microcosm, an idea found, in non-architectural contexts, in the *Republic* of Plato.

§3. Was the demiurge a surveyor?

In contrast to the Christian God who creates the universe *ab nihilo*, the Greek demiurge, epitomized in Plato's *Timaeus*, puts the universe together by constructing order out of chaos. As mentioned above, the word "cosmos" apparently indicated "jewelry" (see Cornford, 1957) and was used by ancient Greek philosophers to express the "beautiful order" of the universe. In the *Timaeus*, the demiurge is pictured as ordering the universe according to certain proportions and constructing the Material Elements from regular solids. Thus, the demiurge is indeed a surveyor, in the wide sense of surveyor/architect indicated in §2, since he delimits the universe by measuring out and apportioning limits. More than this, however, he creates beauty by constructing in accordance with mathematical theory.

§4. Why, then, is there no hint of applications in Euclid's treatise?

The fact that Euclid's *Elements* contains no reference to non-mathematical contexts did not impede the ancients in regarding it as linked to other matters. Thus, Proclus (1992, p. 57) affirmed that

> *... Euclid belonged to the persuasion of Plato and was in home in this philosophy; and this is why he thought the goal of the Elements as a whole to be the construction of the so-called Platonic figures.*

But, perhaps more can be said than is contained in Proclus's vague account. In fact, according to the general view espoused in Plato's *Republic*, mankind's activities are structured by at least three distinct kinds of investigation, to wit, those of mathematics, philosophy and myth. Mathematics is concerned with pure and certain knowledge, whereas philosophy furnishes the hermeneutic principles which make science (applied mathematics) possible; finally, myth presents the result of science in a poetic or dramatic form in order to compel assent from the great majority who have little or no access to mathematics. Plato's own Doctrine of the Divided Line in the *Republic* conforms to this pattern. The mathematical doctrine itself, being part of his esoteric or "unwritten" doctrine, is only hinted at in the *Republic*, but an attempt to reconstruct it can be found in Erickson and Fossa (2006). Some of the hermeneutic principles relating to the application of this doctrine to the regulation of the commonweal are presented in his discussion of the divided line and a mythic account of this same result is proposed in the Myth of the Cave. Consequently, as a theoretical treatise, *The Elements* could not contain reference to applications which are always *a posteriori* and, therefore, uncertain.

§5. What is theory?

Theoretical knowledge is the determination of absolutely true and certain principles and corresponds to the second division (Mathematics) of Plato's Divided Line. In epistemological terms, it is the mode of apprehension called "knowledge" and is completely *a priori*. In ontological terms, its object is non-material reality, since matter undergoes constant change and, therefore, according to this view, cannot be known. In contrast, surveying is a Science and, thus, belongs to the third part of the Divided Line. Since it deals with material objects it is only "opinion", albeit "scientific opinion", and susceptible to the uncertainties of sense perception. Specifically, surveying undertakes to delimit spatial relations on the surface of the Earth. Thus, geometry, as theorizing about surveying, is the determination of the fundamental principles of space itself.

We may further observe that "theory" was also, for the ancient Greeks, the ritual journey to the site of an oracle in order to view the oracle's religious instruments (for more details, see Fossa, 2008). This was supposed to have a transformative effect on the theorizer, but he was not expected to be transported to a more ethereal existence for the rest of his life. Rather, he was expected to return to his native city and put his new insights to work for the benefit of the commonweal. It may therefore be concluded that Euclid's *Elements* was not done *solely* for its own sake, but also for the sake of regulating those sciences to which I have given the collective name of "surveying".

§6. What is mimesis?

In art, mimesis is an imitation of the divine in the form of a representation of it through human speech. It thus makes present to the audience seminal moments of the gods' exploits. In a similar manner, the geometer contemplates the never-changing truths about space and codifies them in human language, thereby making them present to human knowledge. Moreover, just as the demiurge, in his role as divine surveyor, used these mathematical truths to construct the beautifully ordered cosmos, the geometer, in his role as human surveyor, makes possible the application of these same truths in the construction of human projects. Finally, the geometer imitates the divine by acting in a completely rational manner and, hence, in accord with divine rationality.[7]

Conclusion

The present note, by taking a fresh look at the first three postulates of Euclid's *Elements*, opens new possibilities for understanding the role that not only geometry, but, as was mentioned in footnote 6, all of mathematics, played in the intellectual *milieu* of ancient Greek culture. Mathematics, as theorizing, contemplates the absolutely certain and eternal truths that allow mankind to construct his imperfect, although fairly effective sciences,

[7] We may observe that the lusty and querulous Homeric gods are not to be identified with the gods of the philosophers. Plato's demiurge, for example, has virtually nothing, other than immortality, in common with his Homeric counterparts.

when these truths are applied to the material world. This, in turn, implies that, while there was indeed a hierarchy regarding theory and practice, both in terms of the reliability of the mode of apprehension appropriate to each of them and the reality of the objects that each treated of, it was not a hierarchy of polar opposites; rather, it was more like an organic whole, in which each has its part to play in the correct ordering of human activities.[8]

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John A. Fossa
Depto. de Matemática
UFRN – Natal, RN

E-mail: jfossa@oi.com.br

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