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MATHEMATICS AS MEASURE

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There are many different kinds of rules and regulations: mathematical laws, laws of nature, grammar rules, traffic regulations. Though they may all be categorised as laws, they are quite dissimilar. And yet: Might there be a certain correlation between these various law complexes?

According to a modern version of Platonism, mathematical laws are unique in that they apply to a world consisting of immutable mathematical objects. Mathematics expresses laws pertaining to a world of ideas, thus wholly unconnected to laws of nature since these govern objects in the physical realm. This is one way of seeing things. John Stuart Mill thought quite differently on the matter. He believed that mathematical laws are of the same kind as the laws of nature. Mathematical laws, too, express empirical generalisations. The reason why they may be perceived to be absolutely certain and immutable is that they are based on more extensive empirical data than any other laws of nature.

Neither of the above formulations of mathematical laws lead us toward comparing those laws to man-made conventions. There are different types of such conventions. While traffic laws can be passed or repealed from one day to the next, grammatical laws, for example, express regularities that emerge through a sustained process. Grammar rules are expressive of socially constructed, implicit linguistic regularities. All the same, they may also comprise an element of explicit regulation. As written language was propagated in the Renaissance, there was a need to introduce certain standards for the spelling and inflection of words because language was to be printed. Dialects in various oral traditions made up a wealth of variety from which 'correct speech and writing' were defined. In this way, a language may contain rules developed through agelong practice, as well as more explicitly formulated rules.

Is it possible that the laws of mathematics share certain traits with such linguistic regularities? To answer this question in the affirmative is to at least be open to interpretations of mathematics as a social construction.

One first step towards such an interpretation is to see mathematics as coming into existence through the joint efforts of the community. This account of the construction process reflects the fact that even the most brilliant mathematician could not have undertaken it alone; and that new mathematical undertakings are almost always made by several mathematical circles independently and simultaneously. In short, there are social processes involved in the very construction of the mathematical objects with which we occupy ourselves.

Lakatos, for instance, would be inclined to say that mathematicians jointly move towards the best possible theory about mathematical objects and rules of inference and deduction – that it is possible to get closer and closer to the truth of these, even though any postulation about the mathematical world is, in principle, always liable to be false knowledge.

In contrast to this conception, we consider an interpretation of mathematics that more radically places humans at the heart of the proceedings. According to Lakatos, the human community is the constructor of mathematical objects. Still, with him, as with John Stuart Mill, there is a conception of the mathematical objects as partly independent of humans. Following the later Wittgenstein, we shall see how humans can be placed as the only axis on which mathematics revolves.

Wittgenstein returns

It is customary to distinguish between the early and the later Wittgenstein's conception of the nature of language. The distinction owes to the fact that in the years between 1928 and up until his death, Wittgenstein challenged several central points of his early philosophy. The early Wittgenstein's basic assumption about the principle of operation in language is that our terms and sentences acquire meaning through reference to reality. Language consists of sentences that express relations between words. For language to picture reality, the elementary sentences in language must correspond with the elementary facts in reality. Only on the assumption that such depiction is possible can we explain – the early Wittgenstein has it – how language can carry meaning in our communication. According to the early Wittgenstein, our use of language presupposes that the words and sentences of language have a prior meaning deriving from this correlation with facts in the world.

In many ways, the later Wittgenstein turns against this understanding of language. Whereas the early Wittgenstein held that our use of words and sentences is secondary to their meaning, the later Wittgenstein suggests that it is quite the other way around. His motto is brief and to the point: A word's meaning is its use.

As an alternative to the picture theory of language expounded in *Tractatus*, Wittgenstein does *not* propound another theory of the same universal scope. It is now his opinion that no such universal theory about language can be advocated. Instead of *explaining* how language works, he now wishes to describe the ways in which it works. His central term in this description is 'language game.' Wittgenstein plays on both the sports-related and the child's play-related meaning of the word 'game.' Let us first consider the sports-related meaning. There are many different kinds of games. One may think of ball games: football, handball, tennis, badminton (to the extent that a shuttlecock can be considered a type of ball). One may think of board games: monopoly, draughts, chess. And there is bingo, the pools, lotto. In the meaning of child's play, the word 'game' may refer to dead donkey, blind man's buff, hide-and-seek, and one can play with dolls or toy cars. One may play outdoors or indoors. And we must not forget all the computer games that are now on the market.

Thus, there is any number of different types of games, but is there some essence underlying the fact that we call them all games? We may ask if it is possible to define precisely what a game is? Wittgenstein's point is that this would be a hopeless task. One cannot hope to attain a universal definition of 'game' but must content oneself to find groups of games with certain similarities amongst themselves. Wittgenstein speaks of these likenesses as 'family resemblances.'[1] As such, draughts and chess are akin in that they are both played on a board with 64 squares; while tennis, table tennis, and badminton also share certain traits. But to set oneself the task of defining 'game' in a universal scope is not only a hopeless, but also a pointless endeavour. A concept like 'game' is not to be understood by means of its distinctive references, if any, but through the use that can be made of the word in different connections.

By the same token, the later Wittgenstein holds that any attempt to find a shared characteristics of the term 'language' will be to no avail. He thereby rejects the metaphysics of *Tractatus* in which he consistently referred to 'the language' in the definite singular form.[2] Now instead he speaks of different language games that may be used in different contexts, and that get meaning from the uses made of the them in the respective circumstances. In giving up the search for an 'actual' characterisation of language as a whole, the later Wittgenstein enters upon a non-essentialist course quite contrary to the philosophical tradition of Plato and Socrates who directed their philosophy toward finding the essence of the things (their underlying idea). According to Wittgenstein, language and words contain no such underlying idea. Instead we find them in a variety of specific contexts. Understanding of a particular language game is achieved through participation in the practice characterising the game. This is where one may observe and show others how words, sentences, symbols, etc. are *used* in connection with the practical human activity in question in this particular language game.

What does it mean to 'understand' what sort of a thing a hammer is? Perhaps a peculiar question, but to answer it sensibly we might say, in accordance with the later Wittgenstein, that understanding a hammer is closely connected to understanding where, when, and how it is used. One must be able to handle a hammer in some situations and to put it aside in others. To understand a language game is to be able to get on in it.

Man's Mathematical Calculations

How are we to understand all this in relation to our conception of mathematics? Firstly, it must be pointed out that mathematics can be regarded as a form of language. Or rather as a conglomerate of many different language games which may however display strong family resemblances. In other words: we ought not to look for the essence of mathematics. Mathematics more accurately manifests itself in many different forms of mathematical language games. We must examine and attempt to handle those games in their practices in order to achieve an understanding of mathematics and of mathematical

[1] In Wittgenstein's words: "I can think of no better expression to characterize these similarities than "family resemblances"; for the various resemblances between members of a family: build, features, colour of eyes, gait, temperament, etc. etc. overlap and criss-cross in the same way. – And I shall say: 'games' form a family." (Wittgenstein 1983, p. 32e [67]).

[2] Thus, he says that "It is interesting to compare the multiplicity of the tools in language and of the ways they are used, the multiplicity of kinds of word and sentence, with what logicians have said about the structure of language. (Including the author of the *Tractatus Logico-Philosophicus*.)" (Wittgenstein 1983, p. 12e [23])

regularities. Based on Wittgenstein's conception of language games, it becomes clear that the only supplier of meaning to mathematical symbols, formulas, proofs, etc. is the mathematical practice. It is exclusively the use we make of these that gives them meaning. How, then, do these particular mathematical language games look, and what distinguishes them from other language games about cooking, card games, physics, etc.?

What is special about the mathematical language games connected with mathematical research is that they are the ones within which we construct 'proofs.' In keeping with his general philosophy, Wittgenstein claims that there is no essence to be found in the concept of a 'proof.' 'Proof' is a family resemblance concept. It has no essential meaning that we must uncover. Instead it has meaning through the use made of 'proofs' in the group of language games customarily known as mathematics. Every new proof can be said to add to the meaning of the concept. No predefined set of syntactic or semantic conditions for argumentation is necessary for something to count as a proof. Euclid's construction proofs are on the same epistemological level as the proofs in modern formal systems. The multiplicity of different types of proofs in mathematical argumentation is, to Wittgenstein, analogous to the diversity of uses we make of language generally in everyday life: "I should like to say: mathematics is a MOTLEY of techniques of proof. – And upon this is based its manifold applicability and its importance" (Wittgenstein 1978, p.176 [III-46]).

Wittgenstein specifies the motley of techniques as different types of 'calculations.' Mathematical objects such as equations, functions, etc. do not refer to anything in nature or in a Platonic ideal world. Instead they get their meaning from the rules we ascribe to them and in accordance with which we use them. In other words, mathematical objects get their meaning from the way we 'calculate' with them. By 'calculation' Wittgenstein refers to a certain procedure by which we manipulate mathematical objects, e.g. the derivation of one equation from another in accordance with certain rules. Hence Wittgenstein can maintain that mathematics "consists entirely of calculations" (Wittgenstein in Wrigley 1986, p.186).

But why is Wittgenstein so insistent that mathematics is to be regarded solely as calculations? First and foremost, he finds that our conceptions of language as well as of mathematics are full of ways of thinking that reach all the way back to Plato's teachings. According to Plato, the world and its objects as we experience them are merely shades of an ideal world, where the true essence of things are. Behind every word or number in our language, there is an underlying idea or essence to which signs in language and mathematics refer. Wittgenstein turns against this to his mind metaphysical conception and says that language has enchanted us. We believe that there is some hidden meaning behind our words and mathematical symbols, a meaning that said words and symbols are meant to help us uncover. But in Wittgenstein's view, there is no such idea or meaning behind words, nor numbers. All mathematics is about is our manipulation of man-made symbols.

The meaning of mathematics stems exclusively from our use of the symbols, and any talk of reference to some original mathematical ground like the pre-linguistic intuition of the intuitionists or the ideal world of the Platonists, Wittgenstein rejects. Even when the so-called logicist programme in philosophy of mathematics attempted to base mathematics on the certainty that logical deductions – tautologies – represent, that was merely masked Platonism! Wittgenstein is convinced that the logicist programme sees preternatural elements of meaning lurking behind mathematical and logical symbols. He discards the idea that such symbols should have any meaning that is not one hundred percent rooted in our usage. Wittgenstein can pose the question, "In what sense is logic something sublime?" (Wittgenstein 1983, p.42e [89]). The answer is, of course, "In no sense."

Is Logic Something Sublime?

To persuade us that mathematics is an entirely human affair, Wittgenstein must show how logical deductions can be understood within this view. The first step is to demonstrate why it is nonsensical to speak of a form of logical reality with which logical deductions must comply a priori (Wittgenstein 1978, p.40 [8]). In denying logic a sublime status, he breaks with the so-called formalist school in a broad sense. The formalist school sees mathematics as something that can be developed mechanically. A certain point of departure for mathematics is agreed on and after that, everything proceeds automatically regardless of what humans otherwise do or think. This mechanical understanding of mathematics which was formulated by Leibniz, among others, is a main aspect of the formalism that Wittgenstein repudiates. The mechanical represents precisely that which is non-human. The notion of mechanical development points toward an interpretation of mathematics as beyond human influence. As Poincaré has noted, this conception of mathematics does not leave room for genuinely new mathematical findings. The automatic process of mechanics has always been ahead of human insight. If mathematics consists merely of what tautologically follows from a certain point of departure, we are done with actual novelty from the very beginning.

To the later Wittgenstein, mathematics has nothing to do with tautologies whatsoever, and the notion that formulas – in some cases chalk marks on a blackboard – should suddenly start to behave determinedly, having their own course of development no matter what we do in front of the blackboard, goes completely against his fundamental views. The idea that mathematical inference is tautological entails a meaning behind mathematical symbols that does not originate in the human practices involving them. In Wittgenstein's view, the meanings attached to our symbols and to our rules of inference are determined solely by our *use* of them. Nothing that mathematical thinking must observe is hidden under the surface, in the form of universal logical structures or the like. Therefore it is misguiding to think of mathematics as something that *mechanically* follows self-explanatory rules.

Instead of the mistaken mechanical concept of tautologies, Wittgenstein talks about logical deductions and mathematical calculations as 'rule-following activities': using a certain rule in a given situation within a particular language game: "There is nothing occult about this process; it is a derivation of one sentence from another according to a rule; … " (Wittgenstein 1978, p.39 [I-6]).

One senses the ill-concealed sarcasm aimed at whoever believes that the mechanics behind the derivation lies outside the human domain. Wittgenstein holds that deduction is the transformation of an expression into another in accordance with the rules that govern the language game within which the transformation is made (Wittgenstein 1978, p.41 [I-9]).

This means that mathematical deductions and inferences are about adequate training in the use of the inference rules that are accepted practice within mathematics. For this mathematical practice is the only thing that can lend the rule any substance. For instance, the rule 'add 2' does not in and off itself hold a formula for how it is to be applied. At one point, Wittgenstein imagines a school boy splendidly adding 2 up through the sequence of numbers 2, 4, 6, 8, …, and his teacher thinking the boy has understood the rule. But once the boy reaches 1000, he suddenly continues, 1004, 1008, 1012, …. Obviously, he has not understood the use we make of the rule, since he felt it was natural to continue as he did once he reached the four-figure numbers. The point Wittgenstein is trying to make here, about what makes us follow mathematical rules in the way that we do, can be extended. Even the series, 1, 1, 1, 1, 1, … does not in and off itself explain how it is to be understood and thereby continued. Even though, given our mathematical training, we are strongly inclined to continue with the figure one, we might just as well interpret the rule used here as 1, 1, 1, 1, 1, 2, 2, 2, 2, …. In short, Wittgenstein's point is: the meaning of a rule – be it a logical deduction or a mathematical calculation – is its use in the language game, and this use is the limit of our explanations as to why the rule must be followed in the way we usually do. *No* deduction can be self-evident and self-explanatory, and the explanation as to why we make it the way we do must therefore ultimately rely on the simple fact: this is what we normally do within this particular language game.

One might object, here, that if mathematics is nothing more than following rules, we could leave it to a machine. And students of mathematics could simply be paced through a great number of formulas in order to be able to reiterate the teacher's rules. A deeper understanding would be superfluous! Is Wittgenstein's anti-mechanical interpretation of mathematics not in fact rather mechanical? Yet, Wittgenstein's key contention is that the rules only have meaning through social practice. There is an "unbridgeable gulf between rule and application, or law and special use", as he himself puts it (Gefwert, 1998, p.207). Nothing can show us how to prove that 'a+(b+1)=(a+b)+1' is a specific instance of 'a+(b+c)=(a+b)+c.' It is only through our mathematical practice that we determine, normatively, what it is to follow a rule correctly. Machines can behave in a rule governed manner if we program them to do so, but they can never make calculations in the sense that Wittgenstein ascribes the word. For they cannot justify their use of the rule, hold it up to tradition, develop and reinterpret the rule in a given situation, etc. To follow rules in a language game, also in the case of a mathematical language game, is about doing something for a reason, and this is only possible for beings like humans who possess will and purpose.

Social Practice as Foundation

To recapitulate straight away: Wittgenstein finds that mathematics is a normative, rule governed practice. By no means a modest assertion. Through his analysis of rules as patterns in a social practice, he in fact establishes a consistent 'human' interpretation of mathematics. In other words, "The mathematician is an inventor, not a discoverer." (Wittgenstein 1978, p.99 [I-167]).

Wittgenstein even refrains from upholding that the natural numbers exist or refer to anything independently of our mathematical language games. We must resist that kind of unfounded but ever present temptations to idealise the words and numbers in our language.

> Counting (and this means: counting like this) is a technique that is employed daily in the most various operations of our lives. And that is why we learn to count as we do: with endless practice, with merciless exactitude; that is why it is

> inexorably insisted that we shall all say "two" after "one", "three" after "two" and so on. – But is this counting only a use, then; isn't there also some truth corresponding to this sequence?" The truth is that counting has proved to pay. – "Then do you want to say that 'being true' means: being usable (or useful)?" – No, not that; but that it can't be said of the series of natural numbers – any more than of our language – that it is true, but: that it is usable, and, above all, it is used. (Wittgenstein 1978, p.37-38 [I-4])

Wittgenstein thus distances himself from the conception of numbers as somehow corresponding to reality. On the other hand, his standpoint is not one of pragmatism. He stresses that what we actually *do* is the ultimate foundation of our mathematics. Mathematical calculations are techniques embedded in human practices and thereby woven into a miscellany of human language games, and they are to be considered a part of the human way of life. Mathematical calculations and proofs are thus not completely separable from our way of life in other respects – to do mathematics is part of being human.

To Wittgenstein, then, mathematics is fundamentally a social and cultural undertaking. The social aspect is in evidence in that it is our concord in the use of mathematical symbols that leads to our feeling compelled to infer as we do in mathematics. The cogency of logical inference is of a sociological nature (and not of a formal-mechanical nature). According to Wittgenstein there is no deeper explanation of the foundation and certainty of mathematics than exactly this social agreement on the use of mathematical calculations. The fact that social groups (mathematical communities) can establish rules and agree on how these are to be followed is the very groundwork of communication. The question, 'is mathematics absolutely certain?' is thus on a par with 'are grammatical conventions absolutely certain?'

Mathematics as Forms of Representation

Mathematics is an entirely inter-human matter, according to Wittgenstein. Still, is not reality something we can 'strike upon' in some sense? Not only do we feel that mathematical truth is irrefutable (to which Wittgenstein remarks that this cogency is of a social nature); we also feel that such truth is *about* something. That 612 + 5344 = 5956 is to be considered a mathematically true proposition, but also that this proposition is about physical objects: if we add 612 objects to 5344 objects, we will have 5956 objects in total. In what sense is such a simple application of mathematics to be understood as rule-following? For the rules in mathematics do apply to the world outside mathematics as well.

According to Wittgenstein, mathematics serves as a measure by which we can describe our surroundings, and this measure is, to some extent, arbitrary. There is no structure in reality that compels us to develop mathematics in a certain way. For the same reason, there is not one 'true' or 'correct' mathematics. Intuitionist mathematics which disallows certain inconsistency proofs, and classical mathematics which allows them, are both fully valid mathematical systems in Wittgenstein's interpretation. They are simply two separate frameworks for describing the world. Other examples of perfectly acceptable systems might be a mathematics which does not include imaginary numbers, or a mathematics which counts thus: '1, 2, 3, 4, many.' Wittgenstein's point is that even though the latter may appear limited, it cannot be deemed less true than other mathematical systems.

For this reason, there is no basis for distinguishing between vast mathematical systems like *Euclid's Elements* and *Principia Mathematica* in relation to some form of extra-mathematical measure for true theories. How, then, does Wittgenstein explain the development that mathematics has undergone? Wittgenstein's ideas about the origin and growth of mathematics are about our freedom to invent new rules to follow. Sometimes this consists in constructing new rules between 'old' mathematical concepts; sometimes it is about inventing altogether new mathematical concepts or systems. In mathematics, new rules are constantly formed which expand on the old mathematical network of rules and concepts (Wittgenstein 1978, p.99). The construction of complex numbers exemplifies this. Wittgenstein stresses that the construction of new mathematics should be understood as the invention of new forms of representation: "What I want to say is: mathematics as such is always measure, not thing measured." (Wittgenstein 1978, p.201 [III-75]).

Mathematical systems provide us with a standard for representing and describing things in the world. They are measures in the sense that they indicate the rules for language games describing the world. To exemplify such rules for the description of reality, Wittgenstein mentions Einstein's use of the geometry of Bolyai and Lobachevsky. Einstein's use of this non-Euclidean geometry can be seen as the use of an alternative system of mathematical rules that determines how we can describe reality (Shanker 1987, p.270). Thus, mathematical systems are to be regarded as different structures or grids by which we can measure reality. According to Wittgenstein, there is always a given measure involved in descriptions of reality, and particularly in the sciences mathematics fills this role as measure.

But given that mathematics is a measure, it would still have to be organised sensibly in order to be fit for measuring reality? Are there no constraints as to the crazy mathematical ideas and systems we may concoct? Is the development of mathematics really quite arbitrary? Wittgenstein comments on the development of mathematics in the following passage:

> But then doesn't it [mathematics] need a sanction for this? Can it extend the network arbitrarily? Well, I could say: a mathematician is always inventing new forms of description. Some, stimulated by practical needs, others, from aesthetic needs, - and yet others in a variety of ways. And here imagine a landscape gardener designing paths for the layout of a garden; it may well be that he draws them on a drawing-board merely as ornamental strips without the slightest thought of someone's sometime walking on them. (Wittgenstein 1978, p.99 [167])

Wittgenstein's answer is that the development of mathematics is arbitrary in the sense that there is nothing in reality that forces us to develop mathematics in the way we do. On the other hand, we are always guided by something in the act of developing new mathematics. Established tradition may guide our mathematical constructions, and their use in the natural sciences or elsewhere often generate new forms of mathematical measure. So we do have ample reason for constructing mathematics in certain directions, but this is not to say that such reasons must somehow be justified on the grounds of a correlation between reality and mathematical forms of representation. Mathematical forms of representation are autonomous in the sense that their meaning content consists solely in our use of the grammatical rules governing the mathematical language game. If 'measurements of reality' imply that we develop mathematics in new ways, these new ways would not, in Wittgenstein's view, be truer measures or forms of representation; they would simply be new ones indicating new ways of describing reality.

The Relation between Proof and Experiment

To regard mathematics as calculation is a radically new approach to the question of what is expressed by a mathematical proposition. Wittgenstein considers it misguided to think of mathematics as a collection of mathematical propositions that have meaning in themselves, since this easily leads us to think that these propositions were somehow there before we constructed them. He emphasizes that the nature of mathematics is a range of different calculation techniques, rather than a collection of true propositions (Wittgenstein 1978, p.365 [VII-8]). Not surprisingly, this means that Wittgenstein does not consider a mathematical proposition to be self-explanatory. He maintains that to understand the content of a mathematical proposition is to understand the proof of the proposition. Wittgenstein argues thus: "what a mathematical proposition says is always what its proof proves" (Gefwert 1998, p.176). In an illustrative analogy he talks about how a mathematical proposition is attached to its proof the way that the surface of a figure is attached to the figure itself.

To clarify these views we must keep in mind that Wittgenstein differentiated between two different types of proof within the 'motley of techniques' and also between mathematical proposition and conjectures. To Wittgenstein, the crucial point is that in order to be meaningful, a mathematical proposition must belong to a mathematical system of proofs with rules that determine the use of the proposition within the system. This argument parallels his conception that a sentence in natural language only has meaning through the use made of it within a given language game.

Proofs are divided into those that operate within an existing axiomatic system, and those that are constructed to create a new axiomatic system (Shanker 1987, p.83). Mathematical propositions are defined as being part of a mathematical system within which we know how to calculate, and where we are able to prove the mathematical proposition in question. If we have no proof of a given mathematical proposition – as was the case of Fermat's last theorem until recently – it is in fact incorrect to call it a proposition. According to Wittgenstein, it would be more appropriate to call it a 'conjecture', since we have yet to set up the rules in our mathematical language game, indicating how to calculate with and use the proposition in our handling of the symbols. It is simply not clear how we can further calculate on the basis of a conjecture. Solutions for a mathematical conjecture can be approached in two different ways: either by constructing a new rule within an existing mathematical system, or – as sometimes happens – by constructing an entirely new system (Shanker 1987, p.106). The former approach is exemplified in a proof in *Euclid's Elements* in which a new concept was introduced. The latter might be illustrated by some of the first proofs in non-Euclidean geometry where the axioms differed from the ones postulated by Euclid.

With these distinctions Wittgenstein tries to show how a mathematical conjecture is a kind of *stimulus* for our constructions, and not a meaningful proposition requiring a proof of its truth or untruth.

> A mathematician is of course guided by associations, by certain analogies with the previous system. After all, I do not claim that it is wrong or illegitimate if anyone concerns himself with Fermat's Last Theorem. Not at all! If e.g. I have a method for looking at integers that satisfy the equation $x^2 + y^2 = z^2$, then the formula $x^n + y^n = z^n$ may stimulate me. I may let a formula stimulate me. Thus I shall say, here there is a stimulus – but not a question. Mathematical problems are always such stimuli. (Wittgenstein, taken from Shanker 1987, p.113)

The proof gives meaning to the resulting proposition, and it would be misleading to say that the proof has changed the meaning of the conjecture. The conjecture is meaningless since it has no place within the meaning-making frame of a mathematical system. It is a stimulus and not a question, since a question presupposes that we have a method for answering it. Fermat's last theorem was a stimulus to us all when we associated it with the instance where n=2. But only after Andrew Weyl's proof can we talk about the conjecture as a meaningful mathematical proposition, in Wittgenstein's opinion.

We have already touched upon Lakatos' attempt to show that mathematics can be considered a (quasi-) empirical science on a par with physics, chemistry, etc. According to Lakatos, mathematics makes use of experiments in the same way as do the natural sciences. But Wittgenstein turns against this idea and argues that mathematics contrasts with the experimental sciences. The close connection between the mathematical proof and the appertaining mathematical proposition is what characterises the mathematical language games. The natural sciences each have their independent objects to study and experiment with, while mathematics is where we work on the 'grammatical rules' with which to describe the world in our numerous other forms of language games.

> Let us remember that in mathematics we are convinced of grammatical propositions; so the expression, the result, of our being convinced is that we accept a rule. I am trying to say something like this: even if the proved mathematical proposition seems to point to a reality outside itself, still it is only the expression of acceptance of a new measure (of reality). Thus we take the constructability (provability) of this symbol (that is, of the mathematical proposition) as sign that we are to transform symbols in such and such a way. (Wittgenstein 1978, p.162-63 [III-26-27])

The construction of a proof in mathematics convinces us of the truth of the proposition, but it does so in a normative fashion, namely by our acceptance of a new measure of the world – a new measure we are to use. This acceptance determines what is meaningful to say and what is not. Doubt has been ruled out in connection with the mathematical proposition through the normative acceptance of the proof.

Because the mathematical proof is normative in this manner, it can never be refuted by way of experiment. It is in fact nothing to do with empiricism, but with norms for measures and correct inferences in our scientific language games, among others. The mathematical language games are thus of a different nature than those of the other natural sciences:

> We feel that mathematics stands on a pedestal – this pedestal it has because of a particular role that its propositions play in our language games.

> What is proved by a mathematical proof is set up as an internal relation and withdrawn from doubt. (Wittgenstein 1978, p.363 [VII-6])

Wittgenstein thus maintains a sharp distinction between proofs in mathematics and experiments in natural sciences. Proofs are characterised by being the particular type of technique from which doubt is logically excluded.

On the basis of the ideas above, Wittgenstein concludes that a method in mathematics must be fully 'surveyable' for it to be a qualified proof. Since proofs are of a normative character it must necessarily be possible for us to scrutinize a proof step by step in whatever degree of detail we might wish. The proof must be surveyable to us in the sense that we can see clearly every step in the proof employing this or that rule. It needs not be intuitively clear to us in its entirety to be a proof. But if we could not go over everything carefully, it would simply mean that there was no actual proof to convince us, and it is the very ability to convince that is the normative role of the proof (Wittgenstein 1978, p.170-71,173, [III-39,42]). To shed further light on the difference between proof and experiment, we may consider Apel and Haken's famous proof of the Four Colour Theorem. Wittgenstein would not have acknowledged this as a proof since it does not live up to his criteria for mathematical proofs. Apel and Haken used a computer to produce a probability-based proof which cannot be examined by humans (Shanker 1986). According to Wittgenstein, there is nothing to prevent us from using such empirical methods in mathematics for inspiration and stimulus, but they can never serve as proof since they are 'unsurveyable.' Since they are unsurveyable, they cannot in principle exclude doubt as to the validity of the proof:

> I should like to say that where surveyability is not present, i.e. where there is room for a doubt whether what we have really is the result of this substitution, the proof is destroyed. And not in some silly and unimportant way that has nothing to do with the nature of proof. (Wittgenstein 1978, p.174 [III-43])

But how can Wittgenstein say that some methods in mathematics are disqualified as proofs when he considers proof to be a concept of family resemblance? As Wittgenstein sees it, a proof does not need to meet any given criteria of rigorousness for it to be valid. The proof techniques employed by Euclid in his *Elements* are every bit as valid as the techniques in a modern, formalised proof (Wittgenstein 1978, p.164 [III-29]). However, unsurveyable probability methods are ruled out as part of mathematical proof because a proof is a grammatical construction that logically excludes any possible doubt about its validity. The proof cements the construction, making it indubitable. Therefore it needs to be possible to survey every single step of the proof on the basis of the grammatical rules. Methods like the one used by Apel and Haken thereby go against the use we make of the concept of mathematical proof. Mathematics has a function of utmost importance in our lives; one which it does not share with the empirical sciences: it is where we are convinced of what measures to use on the world.

Certain or Fallible Knowledge?

It is clear from what we have seen that Wittgenstein, like Lakatos, wished to describe mathematics as a social construction, but this can mean more than one thing. Whereas Lakatos undertook to show that all mathematical knowledge is fallible, just as the theories of physics are fallible, Wittgenstein is of the opposite opinion. In physics experiments and observations form an important part of the scientific work, but in mathematics they are irrelevant, as Wittgenstein sees it. The result of 2+2 is not dependent on the number of times we have seen a 4 as the result of the calculation. 2+2=4 is not something we have discovered, but rather a form of representation that we agree on how to use, and only has veracity within this usage. Because mathematics is in this sense independent of empirical circumstances, it is theoretically impossible to doubt mathematical propositions in the way that we may doubt the theory of relativity, for example. As Wittgenstein puts it, in physics the presence of doubt is part of the game. In time, observations may be made that compels us to modify or even discard a physical theory. The mathematician, however, is not a discoverer, but an inventor, according to Wittgenstein. Changes in mathematics can only occur if we ourselves construct new mathematical structures and abandon the old. For example, Pythagoras' theorem can never be proved to be 'false' in the ordinary sense of the word. It is simply among our basic means by which to represent reality, and it has been proven within the framework of our present mathematical system. It is a cultural product, and those are quite sturdy and durable. Carrying out his philosophical investigations of mathematics Wittgenstein does not thereby aim to present a secure foundation for our mathematical knowledge, since mathematics is part of our way of life and cannot meaningfully be talked about in terms of true or false. Instead he seeks to weed out any form of metaphysical interpretation or metaphorical notions of mathematics from the existing practices of calculation that make up mathematics.

> Philosophy may in no way interfere with the actual use of language; it can in the end only describe it. For it cannot give it any foundation either. It leaves everything as it is. It also leaves mathematics as it is, and no mathematical discovery can advance it. A "leading problem of mathematical logic" is for us a problem of mathematics like any other. (Wittgenstein 1983, p.49e [124])

Only when we have accomplished a separation of the actual practice in mathematics and e.g. Platonic ideas about it can we achieve a more lucid understanding of what mathematics is. The task of the philosopher is solely to reveal such misunderstandings surrounding the mathematical practice, and not to make interpretations about specific mathematical problems, nor to make suggestions about logical revisions. One might call it a therapeutic commission. If we manage to eradicate the platonic conceptions and other misconceptions about mathematics from the language usage concerning the mathematical practice, nothing remains to be done. Wittgenstein labels such misconceptions 'prose' and seeks to make a distinction between this and the mathematical calculations as they are – the rule-following that mathematicians actually carry out..

Wittgenstein anchors mathematical laws in rule-following, which can be interpreted as an expression of a social practice. In that sense he establishes a parallel between mathematical laws and a sort of grammatical law, if indeed a very robust version. This does not, however, lead him to accept scepticism as an inherent part of an understanding of mathematics. Lakatos' analyses point toward an empirical turn in mathematics, and empiricism immediately occasions a scepticist interpretation of mathematics. But Wittgenstein's social anchoring of mathematics does not have a similar slant, and he attempts to avoid any gradual descent into empiricism.

Linguistic rules may well result in absolute statements. The complete certainty as to all bachelors being unmarried does not rest upon particularly comprehensive and thorough observations concerning bachelors as well as unmarried men. The truth of the sentence 'all bachelors are unmarried' rest solely on grammatical matters. In a similar fashion, we may conceive of mathematical certainty as expressive of linguistic convention: this goes for arithmetical issues, as for geometrical propositions. All are based on conventions regarding use of numbers, and of terms like point, line, plane, etc. There is nothing extraordinary about the fact that new conventions can be established. There is also nothing extraordinary about the fact that propositions can hold good with absolute certainty within a given set of conventions – a given mathematical language game. And different mathematical language games may even be governed by different conventions and laws. Mathematics, according to Wittgenstein, is not mystical and certainly not metaphysical.

Wittgenstein characterises mathematical practices in terms of rules and rule-following. One might remonstrate that his analysis represents mathematics as a stylised and insipid practice. At first glance, one might expect mathematics to be something more than just 'rule-following.' We do consider Wittgenstein's analysis of rule-following a rather narrow, if nevertheless ground-braking opening for a social interpretation of mathematics. In one respect much more radical than Lakatos' understanding. For even though Lakatos gives a remarkable description of the common practice taking place in the proof-counterproof-proof process, it still represents a principle of development within a logical universe as opposed to human culture. Wittgenstein places the mathematical principle of development itself in a social practice. Through his analysis of the mathematical language games, Wittgenstein moreover makes it possible to see mathematics as part of a greater social arena in which economy, ideology, scientific character, cultural diversity, and much more gain significance for our understanding and use of mathematics as a part of the human way of life.

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