

SOME REMARKS ON CONCEPTUAL HISTORY OF MATHEMATICS

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(aceito para publicação em janeiro de 2026)

Abstract

In this work, we consider one of the possible ways of studying history of mathematics which in our understanding should be of primary interest to the working mathematician and to the advanced student of mathematics. We illustrate these ideas with an example and briefly account on recent trends on institutional developments.

Keywords: Conceptual History of Mathematics, History of Mathematics, Epistemological Analysis, Vector Spaces, Tensor Products.

[ALGUMAS OBSERVAÇÕES SOBRE A HISTÓRIA CONCEITUAL DA MATEMÁTICA]

Resumo

Neste trabalho, consideramos uma das possíveis maneiras de estudar a história da matemática que, em nosso entendimento, deveria ser de interesse primordial tanto para o matemático atuante quanto para o estudante avançado de matemática. Ilustramos essas ideias com um exemplo e apresentamos um breve relato sobre as tendências recentes nos desenvolvimentos institucionais.

Palavras-chave: História Conceitual da Matemática, História da Matemática, Análise Epistemológica, Espaços Vetoriais, Produtos Tensoriais.

Introduction

André Weil (1906–1998) was one of the most influential mathematicians of the 20th century due to his fundamental contributions to a wide range of mathematical areas. He was one of the founders of the Bourbaki group and also an important historian of this science.

He published a paper titled *History of Mathematics: Why and How* in the proceedings of the International Congress of Mathematicians held in Helsinki in 1978, where he notes at the very beginning of the paper that, regarding the study of History of Mathematics, “the question ‘Why?’ is perhaps superfluous, or should be better formulated as ‘For whom?’” (WEIL, 1978, p. 227).

There are many ways of considering this subject. Taking into account a possible reader, as suggested, it seems reasonable to consider at least three forms of historical exposition of mathematics¹ among many possible others:

- General History;
- Institutional History;
- Conceptual History.

Here, we understand by *General History* that which is intended for the educated reader with an interest in mathematics. As Weil himself remarks, quoting Moritz Cantor (1829–1920), “one may, in dealing with mathematical history, regard it as an auxiliary discipline, meant for providing the true historian with reliable catalogues of mathematical facts, arranged according to times, countries, subject-matters and authors” (WEIL, 1978, p. 228) or, in the words of Paul Tannery (1843–1904), also cited by Weil, a discipline concerned with “the filiation of ideas and the concatenation of discoveries” (WEIL, 1978, p. 230).

Naturally, most texts on the History of mathematics fall into this category, as do some of the classical well-known references (BOURBAKI, 1999; BOYER & MERZBACHA, 2011; KLINE, 1972) or more recent ones like (COOKE 2013, KATZ 1992, STILWELL 2013).

We also include in this line of work texts and papers dealing with biographies of mathematicians, such as the classics (BELL, 1986; DICK, 1981; REID, 1970).

Institutional History is not directly concerned with the evolution of this science but rather addresses important aspects related to the development of mathematics in certain societies or under specific conditions. To cite just a few among the many available examples, we can mention books that discuss the development and consolidation of mathematics in Brazil, such as (AZEVEDO, 2002; FLAMENT, 2003; SILVA, 2003), and many of the papers published in “Revista Brasileira de História da Matemática” (*Brazilian Journal of the History of Mathematics*).

¹ There are also more sophisticated modes presentation that include the interaction between science and society. In this sense, we can mention the concept of *hermeneutics* in history, very well described in (SCHUBRING, Cap. I).

1. Conceptual History

The third style of studying the history of mathematics under consideration is aimed at a very special type of reader: the mathematician himself, whether student, professor, or researcher.

It has often been observed that mathematicians quite frequently show indifference towards the history of their subject. However, considerations on origins, methods, or generalizations can be inspiring and guide the researcher's work.

As Leibniz had already stated:

"Its use is not just that History may give everyone his due and that others may look forward to similar praise, but also that the art of discovery be promoted and its method known through illustrious examples." (WEIL, 1978, p. 227)

Conceptual history of mathematics can be distinguished from general history in the same way as Russell, in one of his books *Introduction to Mathematical Philosophy*, distinguishes Philosophy of Mathematics from Mathematical Philosophy. He considers the latter a discipline that is more mathematics than philosophy. Similarly, Conceptual History, although it includes much of history in the general sense, may be closer to mathematics itself.

Russell's emphasis on this distinction was so radical that the editor of the first edition of his work wonders whether *"this book is out of place in the present Library"* (RUSSELL, 1920, p. vii), as it was published in the *Library of Philosophy Series* by George Allen & Unwin in 1919.²

When the intended reader is a student of mathematics, the emphasis of the exposition should be on the origin of ideas and why they were natural on the given context of their birth. When the expected reader is a professional mathematician, then, so that *"the art of discovery be promoted"*, in Leibniz's words, a deeper analysis is necessary, also highlighting the connection of the subject studied to other areas where it finds applications.

Thus, it becomes clear that Conceptual History of Mathematics is much broader than the more restricted History of Mathematical Concepts; it is a more dynamic activity that must combine the study of concepts with multiple other considerations.

Among many other things, since mathematics is a science that periodically reconsiders its contents and, as consequence, it develops toward increasing levels of abstraction, it may happen that, as concepts become better understood and more general, their initial motivations get lost in the past.

Poincaré highlights the need to research these circumstances as well, in an almost poetic way:

"You have doubtlessly seen those delicate assemblages of silicious needles which form the skeleton of certain sponges. When the organic matter has disappeared, there remains only a frail and elegant lace-work. True,

² An interesting remark is that the text was written while Russell was in Brixton Prison, England, due to his anti-war positions.

nothing is there except silica, but what is interesting is the form this silica has taken, and we can not understand it if we do not know the living sponge which has given its precisely this form. Similarly, the old intuitive notions of our fathers, even when we have abandoned them, still imprint their form upon the logical constructions we have put in their place.” (POINCARÉ, 1958, p.23)

In an essay on what he calls *Epistemological Analysis of the Genesis of the Theory of Vector Spaces*, Jean-Luc Dorier very aptly describes the conditions a text in this discipline must meet, which, in fact, can also be applied to any work on Conceptual History:

“It must not only trace the origins of concepts but also try to understand how each of them evolved, until attaining its highest degree of generality as is known in modern theory. This implies estimating the importance of all the different stages of evolution with regard to the maturity of the concepts involved, as well as to the potential for generalizations they bear at each stage.” (DORIER, 2002, p. 4]

It is clear then, that those who work in Conceptual History of Mathematics must themselves be mathematicians with a broad technical knowledge of the area whose history they aim to describe. In this sense, Tannery asserts:

“There is no doubt at all, he says, that a scientist can possess or acquire all the qualities needed to do excellent work on the history of his science; the greater his talent as a scientist, the better his historical work is likely to be.” (quoter in WEIL, 1978, p. 231)

One might then ask what separates, in this context, the historian from the mathematician when both study past works. Again, according to Weil, just *“their attitudes and motivations”* (WEIL, 1978, p. 235).

2. An Example

To illustrate these ideas, let us consider a concrete example. To satisfy the criteria considered in the previous section, the study of the history of a given concept should answer some questions, such as the following:

- I. When did the concept under study emerge?
- II. What were the circumstances of the time that made its definition natural in that context?
- III. How did this concept evolve through successive generalizations?
- IV. Which are its consequences and its influence on the development of the area to which it belongs?

Recently, the evolution of the concept of *tensor product of vector spaces and modules* was studied in (DOURADO, 2024). To illustrate the ideas above, we will restrict ourselves to some aspects of the theory in the context of finite-dimensional spaces.

Let V be a finite-dimensional vector space over a field K with basis $\{e_1, \dots, e_n\}$.

Although the definition of a tensor product can be formulated intrinsically and under much more general conditions, for the purpose of our example it will suffice to define the *tensor product* $V \otimes V$ as the vector space over K with as a basis the set of symbols:

$$\{e_i \otimes e_j \mid i, j = 1, \dots, n\}.$$

Then, $V \otimes V$ is the set of all elements of the form:

$$\sum_{1 \leq i, j \leq n} u_i v_j (e_i \otimes e_j).$$

It can be easily verified that if V is an algebra, then $V \otimes V$ is also an algebra with the product defined on basis elements by:

$$(e_i \otimes e_j) \cdot (e_h \otimes e_k) = e_i e_h \otimes e_j e_k$$

and extended by linearity.

On the other hand, a similar concept can be defined in a way close to this one: the *exterior product*. Again, this can be done in a very general context but, for the sake of simplicity, we will limit ourselves to finite-dimensional spaces.

Once again, let V be a finite-dimensional vector space over a field K with basis $\{e_1, \dots, e_n\}$. We define the *exterior product* $V \wedge V$ as the vector space over K generated by the symbols $e_i \wedge e_j$ for $i, j = 1, \dots, n$, with the *convention* that

$$e_i \wedge e_j = -e_j \wedge e_i \quad \text{and} \quad e_i \wedge e_i = 0.$$

We note that the definition can be done purely in rigorous mathematical terms. If S denotes the subspace of $V \otimes V$ generated by all vectors of the form $u \otimes v + v \otimes u$ with $u, v \in V$, then the *exterior product* can be defined as the quotient space:

$$V \wedge V = \frac{V \otimes V}{S}.$$

I. *Let us first consider the question of the origin of concepts.*

Although defined in a less elaborate form, both concepts first appeared in the work of Hermann Günther Grassmann (1809–1877).

This author had shown no mathematical talent in his youth and had no specific university training in mathematics. Nevertheless, he eventually became a secondary school teacher of mathematics and, in the years between 1842 and 1843, wrote his main work, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*³, published in 1844.

We will not delve into Grassmann's life and work here, but the interested reader may consult (FLAMENT, 2005).

Grassmann describes his subject as “*a new branch of mathematics*” (KANNENBERG, 1995, p. 9]. Indeed, in the preface to the work, he writes:

“For a long time it had been evident to me that geometry can in no way be regarded as a branch of mathematics like arithmetic or combination theory; instead, geometry relates to something already given in nature, namely space. I had also realized that there must be a branch of mathematics that yields in a purely abstract way laws similar to those that in geometry seem bound to space.” (KANNENBERG, 1995, p. 11)

After a General Introduction where he seeks to explore the concept of Mathematics as a science, considered as excessively philosophical, he moves on to an introduction to the work itself entitled *Survey of the General Theory of Forms*, where he defines the *elementary conjunction* (KANNENBERG, 1995, p. 36) of forms (which he also calls *magnitudes* and which today we would call *elements*), a concept corresponding to what we would call an “operation”. Considering systems where only a simple connection is defined, in a way that seems somewhat strange to the modern reader, he ends up defining, without using this language or an axiomatic presentation, the notion of a *vector space*.

He uses the term “multiple sum” to designate what we would today call a “linear combination” and introduces the concept of linear dependence in a way strikingly similar to that used today (KANNENBERG, 1995, §106). Later, Grassmann defines what we now call an *n*-dimensional vector space, which he calls a *system of n-th order*:

“We call an elementary magnitude of first order dependent on other elementary magnitudes if it can be represented as a multiple sum of them, whereas we call several elementary magnitudes of first order independent if no dependence, in the sense given, exists between them, that is, if it is not possible to represent any one of them as a multiple sum of the rest. Now by an elementary system of n-th order we mean the collection of elements

³ Since this work was originally written in German [18], we will use the English translation by L. C. Kannenberg. We should also mention that there exists a French translation by D. Flament, and a Portuguese version by T. Dourado, D. Flamente, V. Luchetta, and C. Polcino Milies.

dependent on n elements, these n elements being mutually independent.”
(KANNENBERG, 1995, pp. 170–171]

In fact, he introduces the notion of a vector space in a way that would surprise the modern reader: he considers a set of elements, which we will denote here by $\{e_1, e_2, \dots, e_n\}$, which he calls a *system of units*, and the space is the set of all linear combinations (in his language, multiple sums) of the form:

$$a = \sum_{i=1}^n \alpha_i e_i, \quad \text{with } \alpha_i \in \mathbb{R}.$$

He develops the theory to the point of proving what later came to be known as the *Steinitz Exchange Theorem* which is stated and proved correctly without using determinants (GRASSMANN, 1862, p. 9), from which he deduces (as we do now) the invariance of dimension, as well as a clear proof (GRASSMANN, 1862, p. 13] of the relation

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V).$$

Grassmann’s 1844 book (GRASSMANN, 1844) did not have the impact he had hoped for. Years later, in 1862, he published a new version which he entitled *Die Ausdehnungslehre: Vollständig und in strenger Form bearbeitet* (The Theory of Extension: Completely and in Strict Form)⁴ (GRASSMANN, 1862). The subsequent references to his ideas about products are from this second version, where they are more clearly presented.

At the beginning of Chapter 2, *The Product Structure in General*, n. 37, he states the following definition:

“By the product $[ab]$ of an extensive magnitude a with another b I mean that extensive magnitude (or numerical magnitude) that one obtains when one first conjoins each of the units from which the first magnitude a is numerically derived with each of the units from which the second b is numerically derived in a product, the first factor of which is the unit of the first magnitude and the second factor of which is the unit of the second magnitude, then multiplies this product by the derivation numbers with which those units were conjoined, and {finally} adds the collection of products thus produced, that is,

$$\left[\left(\sum \alpha_r e_r \right) \left(\sum \beta_s e_s \right) \right] = \sum \alpha_r \beta_s [e_r e_s],$$

where e_r, e_s are the units from which the magnitudes are numerically derived, α_r, β_s denote the derivation numbers belonging to them, and the sum refers to the different values of the indices r and s .” (KANNENBERG, 2001, p.19)

⁴ Here again, we will follow Kannenberg’s English translation (KANNENBERG, 2001).

Thus, for the product to be well-defined, it will suffice to define the products of units $[e_i e_j]$. In Grassmann's words:

“Thus, should the concept of a particular product be precisely established, then the necessary definitions must be taken into account in that system of units and in these derivations. As soon as these definitions are taken into account, a particular type of product structure results from the general product structure established above.” (KANNENBERG, 2001, p. 19)

To explain his ideas, Grassmann proceeds to consider a space of dimension 2 and discusses the different possible products.

The first possibility he discusses is to imagine the products $[e_i e_j]$ as “units” (i.e., as elements of a basis) of a new space that, in this case, will necessarily be of dimension 4, formed by all elements of the form:

$$a[e_1 e_1] + b[e_1 e_2] + c[e_2 e_1] + d[e_2 e_2].$$

Changing the notation of the new units $[e_i e_j]$ to our modern notation $e_i \otimes e_j$, we realize that he has just defined the tensor product of spaces.

Following this same line of ideas, Grassmann shows that he can define three other types of products. About the second of them, he says:

“Or one could establish one of them, for example $[e_1 e_2]$, (...) and for the rest perhaps use the definitions $[e_1 e_1] = 0$, $[e_2 e_1] = -[e_1 e_2]$, $[e_2 e_2] = 0$.” (KANNENBERG, 2001, p. 20)

Again, replacing the symbol $[e_i e_j]$ with the notation $e_i \wedge e_j$, we see that Grassmann has just defined the *exterior product* giving rise to what is now known as *Grassmann's exterior algebras*.

Later, in 1878, William Kingdon Clifford (1845-1879) replaced Grassmann's rules by the following:

$$[e_i e_i] = 1 \text{ and } [e_i e_j] = -[e_j e_i] \text{ for } i \neq j,$$

defining the now called *Clifford Algebras*. Both types of algebras have applications in many areas of modern mathematics; in particular, Clifford algebras appear together with Grassmann's exterior algebras in differential geometry.

In short, these concepts emerged naturally when attempting to formally explore the various possibilities for defining products of elements in vector spaces. Given the importance of Grassmann and Clifford algebras, the option leading to tensor product seems to have been generally overlooked while exploring this context.

II. *Why was that moment a natural time for these considerations?*

The first half of the 19th century is precisely when a movement toward the axiomatization of algebra took place, especially in England, introducing an increasing degree of abstraction with publications by George Peacock (1791–1858) (PEACOCK, 1833) and (PEACOCK, 1842–1845), Augustus De Morgan (1806–1871) (DE MORGAN, 1837), George Boole (1815–1864) (BOOLE, 1847), William Rowan Hamilton (1805–1865) (HAMILTON, 1837) and (HAMILTON, 1843), and several others.

It can be said that mathematics was mature enough for movements in this direction to occur.

On the other hand, Grassmann was also continuing the work of his father, Justus Grassmann. As he himself explains in the first version of his work:

“While I was pursuing the concept of the product in geometry as it had been established by my father, I concluded that not only rectangles but also parallelograms in general may be regarded as products of an adjacent pair of their sides, provided one again interprets the product, not as the product of their lengths, but as that of the two displacements with their directions taken into account.” (KANNENBERG, 2001, p. 9)

III. *The evolution of the concept.*

We are interested in the notion of the *tensor product of vector spaces*. However, this product was introduced at such an early stage of the theory that, in a way, it accompanies the evolution of the very notion of vector spaces.

Hamilton introduced the notion of *quaternions* in 1843, which quickly found application in mechanics and electromagnetism. Attempting to simplify the methods of quaternions, Josiah Willard Gibbs (1839–1903) eventually developed what we now call *vector analysis*, a term coined by him, starting in 1881⁵.

In Chapter III of his text *Elements of Vector Analysis*, he associates to each pair of vectors u , and v a symbol which he simply denotes by juxtaposition uv and calls a *dyad*. He then considers the set of *dyadics*, which are nothing more than linear combinations of dyads.

Observing that the set of dyads of elements of a basis, as denoted by Gibbs, can be identified with the symbols $e_i \otimes e_j$, we immediately see that he is reconstructing Grassmann’s tensor product.

When the theory of matrices was already reasonably developed, Leopold Kronecker (1823–1891) introduced a product (which we will not develop here to avoid overloading this work with mathematics) that turns out to be the tensor product of a space of matrices with itself. According to Kurt Hensel (1861–1941):

⁵ At the same time, and for similar reasons, Oliver Heaviside (1850–1925) also developed this theory in similar terms.

*“I would also like to point out that Mr. Kronecker had already cited this theorem some time ago in his algebraic lectures and had proved it by an elegant transformation of the determinant under investigation with the help of the multiplication theorem.”*⁶ (HENSEL, 1890, p. 319)

The determinant he refers to is precisely the determinant of the tensor product of two matrices. To this day, there is some debate regarding the priority of defining this concept. See (DOURADO, 2024, Cap.3).

The first general definition, in terms still in use today, dates from 1938 and is due to Hassler Whitney (1907–1989) (WHITNEY, 1938). Although initially stated only for tensor products of abelian groups, the definition holds in the more general context of modules over rings.

IV. *Some consequences and applications.*

To cite just one well-known application, tensor products are used to extend spaces over a given field K to spaces over an extension L of K ; consequently, they find natural application in the problem of extending representations of groups and algebras. See, for example, (CURTS & REINNER, 1962).

Many important results in algebraic topology are formulated in terms of Grassmann algebras, e.g., if G is a finite-dimensional H -space (e.g., a Lie group), the cohomology algebra $H^*(G, K)$ of G with coefficients in a field K of characteristic zero is a Grassmann algebra with odd-degree generators. If G is a simply-connected compact Lie group, then the ring $K^*(G)$, studied in K -theory, is also a Grassmann algebra (over the ring of integers). Not to mentioning the fundamental role that this family of algebras plays in the theory of rings with polynomial identities; see, for example, (ALJADEFF et al. , 2020).

To conclude this section, we would like to note that there are numerous books written in the spirit of Conceptual History. To mention just two particularly interesting ones, we recall Curtis’s text (CURTS, 1999) on the history of representation theory and Flament’s book (FLAMENT, 2003) on the development of complex numbers, which received the *Prix Jean Scott*, *Fondation Jean Scott L’Erigène (Neuchâtel)*, Switzerland, in 1992.

3. Institutionalization

Papers and books on Conceptual History of Mathematics have been written for a long time. However, with the more recent interest in Mathematical Education and, more particularly, in the use of history in teaching this discipline, work in the area has intensified. Since the 1980s, many meetings on the history of specific topics have been organized, such as *Theory of Dimension*, *Foliations and Geometric Quantization*, *Geometry in the 20th Century*, *Geometry and Differential Calculus in the 17th Century*, etc.

⁶ In the original German language: “*Ich bemerke noch, dass Herr Kronecker diesen Satz schon vor längerer Zeit in seinen algebraischen Vorlesungen angeführt und ihn durch eine elegante Umformung der zu untersuchenden Determinante mit Hilfe des Multiplicationssatzes bewiesen hat.*”

Several great mathematicians, such as Michael Atiyah, René Thom, Alain Connes, Pierre Cartier, and Jean Dieudonné, have engaged themselves with the subject. In particular, there is an ongoing collaboration between researchers from France and Latin America that has resulted in the organization of three international meetings:

- *First Franco-Brazilian Colloquium on Conceptual History of Mathematics*, at the Federal University of Rio de Janeiro, in August 2009, organized by D. Flament and held in conjunction with the *First Poincaré Day*, in partnership with the French institutions *Archives Henri Poincaré* and *Fondation Maison des Sciences de L'homme de Paris*.
- *Second School of Conceptual History of Mathematics*, held at the Faculty of Mathematics, Astronomy, Physics, and Computing of the National University of Córdoba, Argentina, in 2010, organized by D. Flament and N. Andruskiewitsch. There are published proceedings of this meeting⁷.
- *Third School of Conceptual History of Mathematics*, at the Institute of Mathematics and Statistics of the University of São Paulo, in Ubatuba, organized by D. Flament and C. Polcino Milies, in 2012.

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⁷ Cf. *Actas de la Segunda Escuela de Historia Conceptual de las Matemáticas*, https://www.famaf.unc.edu.ar/~andrus/Acta%2015_v6.pdf.

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